Directed percolation with long-range interactions: Modeling nonequilibrium wetting

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It is argued that some phase transitions observed in models of nonequilibrium wetting phenomena are related to contact processes with long-range interactions. This is investigated by introducing a model where the activation rate of a site at the edge of an inactive island of length ℓ is $1+a\ell^{-\sigma}$. Mean-field analysis and numerical simulations indicate that for $\sigma>1$ the transition is continuous and belongs to the universality class of directed percolation, while for $0<\sigma<1$, the transition becomes first order. This criterion is then applied to discuss critical properties of various models of nonequilibrium wetting.

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I. INTRODUCTION

Many recent theoretical studies have shown that the growth process of a solid phase on a substrate can undergo a variety of nonequilibrium transitions. They are analogous to equilibrium wetting phenomena in which liquid boundary layers exhibit critical behavior in the vicinity of the liquid-gas coexistence line. Such growth processes can be effectively modelled by defining a suitable evolution rule for the profile (interface), corresponding to the boundary of the solid layer. Several growth models which do not obey detailed balance and evolve towards stationary non-equilibrium states have been studied in the past. In many cases, by varying a control parameter, they exhibit a transition from a regime where the solid phase remains pinned to the substrate to a regime where an unbounded growth sets in [1–8]. This depinning of an interface may be considered as nonequilibrium wetting, in analogy with its equilibrium counterpart. In some models the character of the depinning transition changes, depending on the dynamical rates which control the interface evolution. Numerical studies in $1+1$ dimensions allowed to identify both first- and second-order phase transitions, those of the latter type falling into two universality classes: directed percolation (DP) [9] and multiplicative noise (MN) [10]. Since the transition always occurs towards an *irreversibly* depinned phase, it is quite natural to draw an analogy with dynamical processes characterized by an absorbing state.

Recently, it has also been shown that such a depinning transition can be related to the *synchronization* phase transition in spatially extended chaotic systems [11,12]. In this latter framework, two different replicae of the same dynamical system are coupled one to each other, either deterministically [12] or by the addition of the same realization of a spatiotemporal stochastic noise. Upon increasing the coupling parameter, the system undergoes a nonequilibrium phase transition between an unsynchronized phase and a completely synchronized one, characterized by a vanishing local difference between the two (initially) different replica. Since two completely synchronized replica of the same system are bound to remain identical one with respect to each other at all future times, we can conclude that the completely synchronized phase is absorbing for system dynamics, thus playng the role of the irreversibly depinned phase in nonequilibrium wetting processes. Interestingly, both numerical analyses of coupled map lattices systems [11,12] and analytical arguments [12,13] predict the synchronization transition to exhibit a critical behavior, which belongs either to the MN or to the DP universality class.

A schematic typical configuration of an interface bound to a substrate is depicted in Fig. 1. It is composed of detached domains separated by pinned segments. The dynamics of the interface is such that each detached domain may either shrink or expand from its edges; new domains may be created by the unbinding process of bound sites, and two or more detached domains may merge into a single larger one. In principle, a segment internal to a detached domain may bind back to the substrate. However, in some physical conditions such processes are virtually suppressed. This occurs when the unbound interface moves on the average away from the substrate, while it is held bound to the substrate by some short range attractive interaction. In fact, the farther is a segment from the edge of a domain, the larger is the height of the inferface and, accordingly, the more unlikely the possibility to bind back to the substrate. In this case, the dynamics of the interface may very well be described by a contact process in which the active sites correspond to those bound to the substrate. The resulting depinning transition is thus expected to belong to the DP universality class. The observation of DP depinning transition in some models [1,8,14] confirms the validity of the above arguments. In some cases

FIG. 1. Schematic configuration of an interface bound to a substrate. Within the DP framework bound sites may be considered as active while islands of depinned sites are inactive.

a possible crossover to a first order transition has been suggested, when the attractive interaction between the interface and the substrate is increased. While in certain regions of the phase diagram the existence of a first order transition has been clearly demonstrated, in other regions it has only been tentatively suggested, based on numerical simulations of finite systems [3,14]. Due to the strict connection between wetting and complete synchronization non-equilibrium phase transitions, a more accurate understanding of this part of the wetting phase diagram is highly desirable not only from the theoretical point of view, but also in view of possible experimental realizations of the MN and/or the DP universality classes. In our opinion, in fact, the synchronization transition in spatially extended systems represents the most promising framework in which one can look for MN critical properties, if not for DP ones, which seem to require a highly nonlinear local dynamics governing a spatially extended system [11,13]. It is worth stressing that both these classes are still eluding a clear experimental evidence, possibly due to the presence of quenched disorder in experimental realizations and, in the case of DP phase transitions, to seemingly unavoidable small fluctuations of the (supposedly) absorbing state [9]. Interesting candidates for experiments include the photosensitive Belousov-Zhabotinsky (BZ) reaction, which is known to exhibit complex spatiotemporal dynamics [15] and semiconductor lasers with time-delayed optical feedback. In Ref. [16] it has been shown that such delayed systems can be interpreted in terms of a suitable spatiotemporal dynamics, where the effective system size is given by the ratio between the delay time and the typical fast time scale of the system. Notice that, in principle, the use of semiconductor lasers with time-delayed feedback allows for obtaining rapidly a large data set, which is basically free of quenched noise effects. These features provide the possibility of a proper statistical description of the synchronization transition. Finally, the role played by small fluctuations near the completely synchronized phase and their exponential suppression in the case of synchronization DP has been discussed in Ref. [13].

In the present paper we introduce a framework within which the crossover from DP to first order transition may be examined. This framework is then applied to two previously introduced models of nonequilibrium wetting.

Preliminary simulations performed in the pinned phase close to a seemingly first-order wetting transition have revealed that the activation rate at the border of a depinned island depends on the island length. This suggests that the dynamics of a fluctuating interface leads to an effective interaction between the sites at the island boundaries. If this interaction is long range, it can affect the dynamics of large islands and, in principle, provide a mechanism for a first order wetting transition. In this paper we consider a generalization of the contact process where the activation rate of sites at the boundary of an inactive domain decays algebraically with the domain length. In particular, the activation rate for an island of length ℓ is assumed to take the form $\lambda(1+a/\ell^{\sigma})$, where λ, σ, and a are positive constants. We find that, depending on the power σ, the model exhibits either a continuous DP-like (for $\sigma>1$) or a first-order (for $\sigma<1$) transition. We then examine the possible emergence of such effective long range interactions in wetting models. Numerical determination of the power σ of the effective interaction thus allows inferring the order of the phase transition also in these models.

The paper is organized as follows: In Sec. II we introduce a generalized contact process in $1+1$ dimensions that includes an activation rate which decays algebraically with the size of inactive domains. The mean-field solution of the model predicts a first-order transition, when the interaction decays slowly enough with the inactive domain size, while faster decay rates yield a DP behavior. In Sec. III we present the results of detailed numerical simulations, which are in a very good agreement with the mean field predictions. The bridge between this generalized contact process and wetting models is discussed in Sec. IV. There we consider two models, which have been introduced previously for studying nonequilibrium wetting. The first is a solid-on-solid model [2–4], whose phase diagram contains a line of wetting transitions. While the first order nature of the transition has clearly been established in a part of the phase diagram, the nature of the transition in another region proved to be more difficult to analyze. Our numerical study seems to suggest that inside the latter region, upon changing a suitable control parameter, the effective activation rate at the boundary of inactive domains can exhibit both a fast ($\sigma>1$) and a slow ($\sigma<1$) power law decay. However further analysis based on scaling arguments shows that the slow decay of the effective activation rate (and thus the first order behavior) is only a finite-size effect, albeit particularly robust. This result suggests that in this entire region the transition is asymptotically continuous and of DP nature. We then analyze a second wetting model (single-step-with-wall) [8], where previous studies have indicated a continuous DP-like wetting transition for strong attractive interaction between the substrate and the interface. We provide numerical evidence that the effective activation rate governing the dynamics of inactive islands corresponds to the case $\sigma>1$. This is indeed consistent with the DP nature of the transition. The main results are summarized in Sec. V, which contains also some remarks and comments on future perspectives.

II. A CONTACT PROCESS WITH LONG RANGE INTERACTIONS

In this section we introduce a lattice model of a contact process in $1+1$ dimensions with long range interactions which shows a crossover between a continuous DP and a first-order phase transition. We consider a periodic lattice of length L, where the state variable S_i at site i is either "active," $S_i=1$, or "inactive," $S_i=0$. The dynamics evolves by random-sequential updates, i.e., at each time step a lattice site i is chosen at random. If the selected site is either active or next neighbor of an active site, it is updated according to the following rules

$$1 \to 0 \quad \text{with rate } 1, \tag{1}$$

$$0^{\ell}1\ [10^{\ell}] \to 0^{\ell-1}11\ [110^{\ell-1}] \quad \text{with rate } \overline{\lambda}(\ell) = \lambda(1+a/\ell^{\sigma}),$$

where 0^{ℓ} is a shorthand notation for an inactive island of size ℓ. Finally, as for usual contact processes, inactive sites that

are not adjacent to active ones cannot be activated, thus guaranteeing that the inactive state is absorbing. The constants λ and a are both non-negative: the case $a=0$ corresponds to the usual short range contact process, which exhibits a DP transition at $\lambda_c=1.64892(8)$ [17]. For $a>0$ the power law decay with ℓ of the shrinking rate of inactive islands introduces effective long-range interactions. Within the mean field approximation one finds that the transition is continuous for $\sigma>1$ but becomes first order for $0<\sigma<1$. To demonstrate this point let $\rho(t)$ be the average density of active sites at time t. In the thermodynamic limit, $L\to\infty$, the mean field dynamics of ρ reads

$$\frac{d\rho}{dt} = -\rho + \lambda\rho^2\sum_{\ell=1}^{\infty}\left(1+\frac{a}{\ell^\sigma}\right)(1-\rho)^\ell$$

$$= (\lambda-1)\rho - \lambda\rho^2 + \lambda a\rho^2\sum_{\ell=1}^{\infty}\frac{(1-\rho)^\ell}{\ell^\sigma}. \qquad (2)$$

For $\sigma>1$ the sum on the right-hand side (RHS) of Eq. (2) is finite in the limit $\rho\to 0$ and its contribution amounts to a renormalization of the coefficient of the ρ^2 term. Accordingly, the mean field equation describes a standard DP process with short range interaction, thus recovering, for sufficiently small values of a, the continuous nature of the phase transition. For large a, the coefficient of the ρ^2 term becomes positive and the mean-field approximation predicts a first-order transition. However, studies of similar models indicate that the mean-field prediction of first-order phase transitions associated with the change of sign of the ρ^2 term is unreliable [18].

For $0<\sigma<1$, the leading contribution arising from the sum in the rightmost side of Eq. (2) can be captured by replacing it with an integral over $d\ell$,

$$\partial_t\rho = (\lambda-1)\rho - \lambda\rho^2 + \lambda a\rho^2\int_0^\infty \frac{(1-\rho)^\ell}{\ell^\sigma}d\ell, \qquad (3)$$

which, to leading order in ρ, reduces to

$$\partial_t\rho = (\lambda-1)\rho + a\lambda\Gamma(1-\sigma)\rho^{1+\sigma} - \lambda\rho^2. \qquad (4)$$

Here $\Gamma(x)$ is the standard Gamma function. The leading nonlinear term in this equation involves a noninteger power, as a consequence of the long range nature of the interactions. Since its coefficient is positive, Eq. (4) cannot admit fixed point solutions for arbitrarily small densities, so that the transition to the absorbing state is first order. This result is expected to hold even for $\sigma=1$, where the leading singular term in the equation is $-\rho^2\ln\rho$.

III. NUMERICAL RESULTS

The mean field calculation, that is expected to hold above the upper critical dimension of directed percolation $d_c=4$, indicates that for $\sigma>1$ the transition is second order with the critical exponents of DP, while for $\sigma\leqslant 1$ it turns into a first-order transition. The crossover from a continuous to a discontinuous transition is therefore predicted to take place at $\sigma_c=1$. In what follows we investigate the validity of this prediction by simulating the generalized contact process defined in Eq. (1). In order to obtain independent checks, we performed two different kinds of numerical analyses, i.e., measuring the scaling properties of suitable observables starting from (i) a fully active state and (ii) a single active site (epidemic spreading). Although we do not expect the overall scenario to depend on the parameter a [see Eq. (1)], the accuracy of the numerical simulations actually does depend. For small a, discontinuities in the order parameter are correspondingly small, while for large a, it is necessary to consider very large lattice sizes to reach the asymptotic-scaling regime. All the numerical results reported in this paper have been obtained for $a=2$, which represents a good compromise. We find $\sigma_c=1.0\pm0.1$, which suggests that the mean-field analysis is quite accurate even in $1+1$ dimensions. However, we cannot exclude the possibility that the precise threshold value in such a low dimensional case slightly deviates from 1. This has already been observed in directed percolation with long-range infections through Levy flights, where a small deviation from the mean-field prediction was found for the critical value of the control parameter (i.e., the exponent of the Levy distribution) [19,20].

A. Analysis of the stationary active state

We first discuss the evolution of Monte Carlo simulations that start from a fully active state. In order to distinguish between first-order and second-order transitions we determine the density of active sites and the size distribution of inactive islands. Let us first summarize the expected results for each of the two quantities. Since the continuous transition should be DP, the average density of active sites, $\rho(t)$, measured at criticality, should decay as

$$\rho(t)\sim t^{-\theta}. \qquad (5)$$

On the other hand, off-criticality, in the active phase of an infinite system, $\rho(t)$ saturates to a stationary value $\langle\rho\rangle_t$ (where $\langle\cdot\rangle_t$ denotes time average), which scales with the distance from the critical point λ_c as

$$\langle\rho\rangle_t \sim |\lambda-\lambda_c|^\beta, \qquad (6)$$

where $\beta=0.276486(8)$ [21]. These two critical exponents are connected by the so-called temporal exponent $\nu_\parallel$, so that $\theta=\beta/\nu_\parallel=0.159464(6)$ [21].

Moreover, in DP the size distribution $P(\ell)$ of inactive islands decays algebraically as

$$P(\ell)\sim \ell^{-(2-\beta/\nu_\perp)} \qquad (7)$$

with a cutoff at the spatial correlation length $\xi_\perp\sim|\lambda-\lambda_c|^{-\nu_\perp}$, where the exponent $2-\beta/\nu_\perp=-1.747\ldots$ follows from $\langle\ell\rangle\sim 1/\rho\sim\xi_\perp^{\beta/\nu_\perp}$. In fact, the average length of inactive islands

$$\langle\ell\rangle = \int_0^\infty P(\ell)\ell\, d\ell \qquad (8)$$

diverges at criticality due to scale invariance, so that $P(\ell)\sim\ell^{-\alpha}$ with $1<\alpha<2$ for $\ell\lesssim\xi_\perp$. Since the average size of inactive islands is proportional to the inverse of the density

TABLE I. Estimates of the critical points for $a=2$. They have been obtained starting from a fully active initial state (λ_c^f) and by spreading analysis of localized initial conditions (λ_c^l). Finite size corrections are responsible for the small differences.

σ	0.5	0.8	0.9	1.1	1.2	1.5
λ_c^f	1.2850(5)		1.3650(5)	1.4091(1)		1.4708(3)
λ_c^l	1.2870(4)	1.3395(3)	1.3635(5)	1.4093(3)	1.4280(5)	1.4711(3)

of active sites, one has $1/\rho \sim \langle \ell \rangle \sim \xi_\perp^{\beta/\nu_\perp}$, which implies $\alpha = 2 - \beta/\nu_\perp = 1.747\ldots$ [22–24].

Conversely, for a first-order phase transition $\rho(t)$ is not expected to exhibit any critical behavior associated with a diverging correlation length at the transition point. Instead, the saturated order parameter $\langle \rho \rangle_t$ exhibits a discontinuity at the transition. Since in this case the active phase cannot display any coarsening properties, the average length (8) of inactive islands should be finite, i.e., $P(\ell)$ should decay *faster* than $1/\ell^2$. As no hysteretic behavior can be observed in nonequilibrium processes with an absorbing state, the distribution $P(\ell)$ turns out to be the most effective indicator of a first-order transition.

In order to reduce as much as possible finite-size effects, we considered very large systems of size $L=2^{19}$ with periodic boundary conditions (additionally we have also averaged over a few different realizations). The best estimates of the critical point λ_c^f, are reported in the first row of Table I. For $\sigma=1.1$ and $\sigma=1.5$ we identified a critical scaling region (see Fig. 2), where both the exponents δ and β are in agreement with the best numerical estimates for DP in 1+1 dimensions. Moreover, we computed $P(\ell)$ in the active phase close to the critical point by sampling spatial configurations at periodic time intervals and counting all the inactive regions of size ℓ. As shown in Fig. 3, $P(\ell)$ is characterized by a power law decay slower than $1/\ell^2$ (prior to the unavoidable exponential cutoff due to finite-size effects), consistently with the prediction for DP [see Eq. (7)].

On the other hand, below $\sigma=1$ a scaling region at the transition point could not be identified. The saturated density of active sites $\langle \rho \rangle_t$ shows a finite discontinuity and $P(\ell)$ decays faster than $1/\ell^2$ (see Fig. 3). Accordingly, this analysis provides evidence that the transition is discontinuous.

B. Spreading from a single seed

A further verification of the mean-field analysis has been obtained by simulating model (1), starting from a single active site at the origin [25]. In this type of simulations, the relevant variables are the survival probability $P_s(t)$, the number $N(t)$ of active sites (averaged over all runs) and the mean square spreading $R^2(t)$ of the active region. At the critical point of a phase transition towards an absorbing state, these quantities are known to scale as

$$P_s(t) \sim t^{-\delta}, \quad N(t) \sim t^{\eta}, \quad R^2(t) \sim t^{2/z}. \tag{9}$$

In the special case of a DP phase transition, the exponents δ, η, z can be expressed in terms of the standard exponents $\beta, \nu_\perp, \nu_\parallel$ as [26]

$$\delta = \theta = \beta/\nu_\parallel, \quad \eta = (d\nu_\perp - 2\beta)/\nu_\parallel, \quad z = \nu_\parallel/\nu_\perp. \tag{10}$$

Their actual values are reported in the first line of Table II. A scaling behavior of the type described by Eq. (9) is

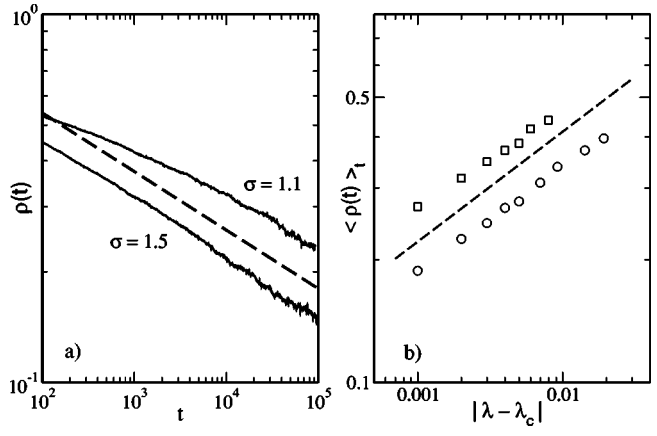

FIG. 2. Power law critical behavior of the generalized contact process (1) for $\sigma=1.1$ and $\sigma=1.5$, compared to the expected DP behavior (dashed lines): (a) decay in time of the density of active sites $\rho(t)$; (b) stationary density of active sites as a function of the distance from criticality. Circles correspond to the case $\sigma=1.5$, while squares to $\sigma=1.1$. Both graphs are plotted in a doubly logarithmic scale.

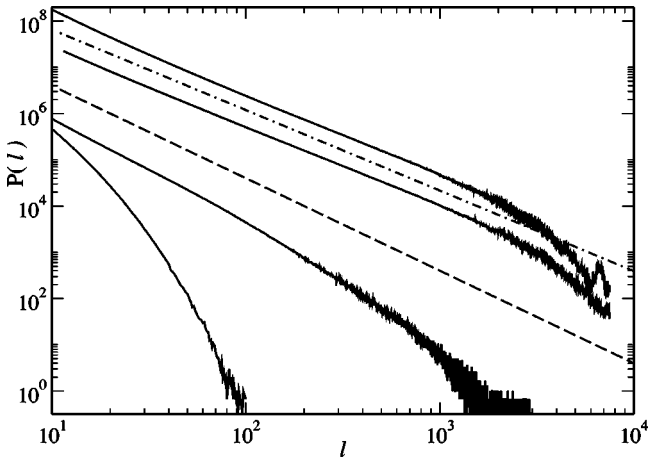

FIG. 3. Doubly logarithmic graph of the (un-normalized) size distributions of inactive islands $P(\ell)$ as a function of size ℓ. From top to bottom the solid lines correspond to $\sigma=1.5$, $\sigma=1.1$, $\sigma=0.9$, and $\sigma=0.5$. The dot-dashed line marks the power-law decay expected in the case of a DP phase transition. The long dashed line decays as $1/\ell^2$, discriminating in 1+1 dimensions between first order and continuous phase transitions.

TABLE II. Exponents in seed simulations for DP and zero-temperature Glauber dynamics in 1+1 dimensions.

	δ	η	z
DP	0.159464(6)	0.313686(8)	1.580745(10)
Glauber	1/2	0	2

expected to arise also in systems exhibiting a first-order transition in 1+1 dimensions [18], although the relations between spreading and stationary exponents assume a more general form than Eq. (10) [9]. In these cases, the critical dynamics follows from the marginal relative stability of two coexisting phases: (i) the absorbing state itself and (ii) a phase characterized by a suitable finite density of active sites. For instance, an active site evolves into a droplet of phase (ii) embedded into a sea of phase (i). At criticality, the boundary, having no preferential velocity, diffuses like an unbiased random walk. A clear example of such a type is the one-dimensional Glauber-Ising model at zero temperature, where the density of active sites in phase (ii) is maximal, i.e., equal to 1. In models such as Eq. (1), the natural existence of small inactive islands (see Fig. 4) can seemingly make the separation into two phases questionable. However, for sufficiently small σ values, the long-range interactions are strong enough to suppress the formation of large inactive islands. In these circumstances, one would expect the long-term dynamics to be controlled by the evolution of the boundaries. In particular, the scaling exponents of contact processes exhibiting a first-order phase transition in 1+1 dimensions are expected to be the same as in the Glauber-Ising model at zero temperature, whose values are reported in the second line of Table II. Notice that $\eta=0$ means that the density remains finite at criticality.

Numerical simulations of the spreading dynamics confirm our expectations. Independently of the results discussed in the previous section, we have first estimated λ_c by measuring the average number of active sites $N(t)$ for different values of λ and then looking for the value of the control parameter that minimized the curvature of $N(t)$ at long times (see Fig. 5). By performing simulations up to $t_{\max}=3\times10^6$ Monte Carlo sweeps (MCs) and by averaging over 10^5 realizations, we have obtained fairly accurate estimates, which are listed in the second row of Table I. The differences with the values reported in the first row provide an indirect estimate of the magnitude of finite-size corrections.

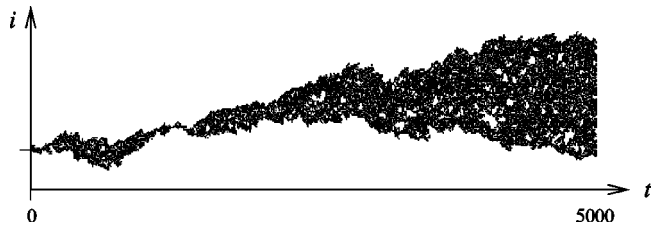

FIG. 4. Typical cluster grown from a single seed for $\sigma=0.5$ at criticality, simulated up to 5000 Monte Carlo sweeps. Notice that large inactive islands are suppressed and the growing cluster can be regarded as effectively compact.

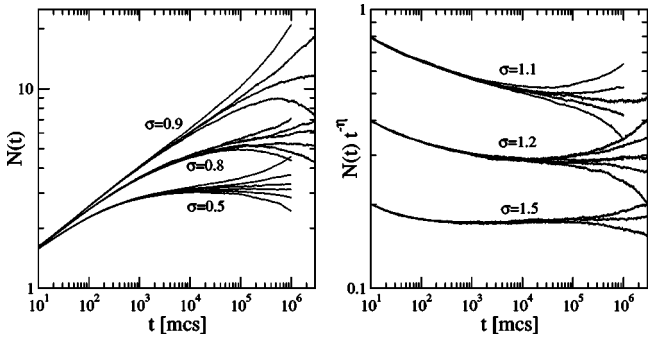

FIG. 5. Time evolution of the average number of active sites close to the phase transition. In the left panel the average number of active sites $N(t)$ is plotted as a function of time for $a=2$, $\sigma=0.5, 0.8, 0.9$ and for different values of λ, close to λ_c. All curves indicate that $N(t)$ converges to a constant after a transient time which increases with σ. In the right panel the time-rescaled number of active sites $N(t)t^{-\eta}$ (with $\eta=0.313686$) is shown for $a=2$, $\sigma=1.1, 1.2, 1.5$, and for different values of λ, close to λ_c. In order to avoid a messy overlap, the curves have been shifted vertically by an arbitrary value. Their asymptotic behavior indicates good agreement with the expected DP critical scaling. The estimates of λ_c are reported in Table I. Both graphs are plotted in a doubly logarithmic scale.

We have also investigated the asymptotic behavior of $P_s(t)$ and $N(t)$ at criticality (see Fig. 6). For large times, both quantities exhibit a power-law behavior: for $\sigma>1$ ($\sigma<1$) the growth rates are consistent with a DP transition (Glauber-Ising dynamics). As expected, the closer is σ to 1, the longer is the time needed to reach the scaling regime.

In conclusion, our simulations indicate that the crossover from a continuous to a discontinuous transition takes place in our model between $\sigma=0.9$ and $\sigma=1.1$. This result is consistent with the predictions of the mean field arguments discussed in Sec. II. For the sake of completeness, we mention the results of simulations made in the marginal case $\sigma=1$. In such a case, there are indications of a still first-order transition, although the Glauber-Ising exponents are not yet recovered on the accessible time scales.

IV. NONEQUILIBRIUM WETTING AS A CONTACT PROCESS WITH LONG RANGE INTERACTIONS

In this section we investigate to what extent the behavior of nonequilibrium wetting processes can be interpreted as a contact process with long-range interactions of the form (1) and whether the previous results can be used as a criterion for distinguishing first-order from DP-like continuous transitions. To this end we consider two previously introduced wetting models, studying them within the above derived framework. In practice, we numerically estimate the effective activation rates at the boundary of detached islands and show that they are indeed of the form (1). The results obtained in the previous section can help to discern the nature of the wetting transition in the two models.

A. Restricted solid-on-solid wetting model

The first system we consider is a restricted solid-on-solid (RSOS) model [2–4]. It is defined on a one-dimensional lat-

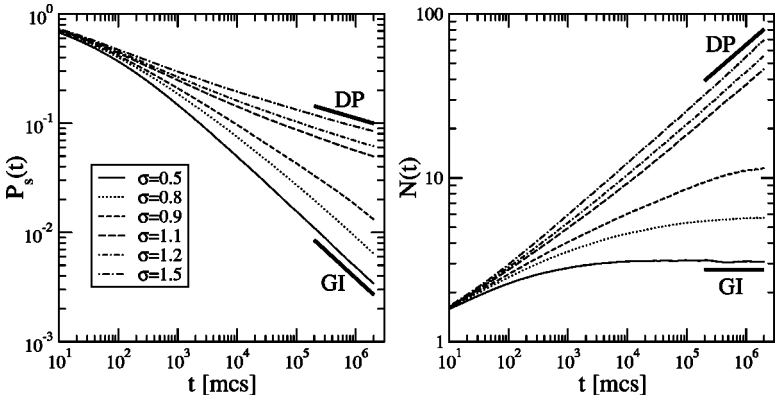

FIG. 6. The survival probability $P_s(t)$ (left panel) and the average number of active sites $N(t)$ (right panel) at criticality for $\sigma = 0.5, 0.8, 0.9, 1.1, 1.2, 1.5$ (shown from bottom to top) in a doubly logarithmic representation. The predicted asymptotic slopes of directed percolation (DP) and Glauber-Ising (GI) are indicated as bold lines.

tice with periodic boundary conditions: at each lattice site i the height variable h_i can take any nonnegative integer value such that $|h_i - h_{i+1}| = 0, 1$. A hard attractive substrate is located at zero height preventing h_i from becoming negative. The interface evolves by random sequential updates controlled by three real parameters q, q_0 and p. At each move, a site i is randomly selected and, provided the above constraints are fulfilled, one of the following three processes is carried out (see Ref. [4]).

Particles are deposited with rate q_0 at the bottom layer and with rate q at higher layers.

Particles evaporate from the edges of a terrace with rate 1.

Particles evaporate from the middle of a plateau with rate p.

The phase diagram of the model is shown in Fig. 7. If q_0 is large enough (e.g., $q_0 \approx q$), the model exhibits a line of continuous phase transitions, which belongs to the MN universality class. Its critical behavior can be described by a Kardar-Parisi-Zhang (KPZ) equation [27] in a potential $V(h)$ representing the interaction between the interface and the substrate

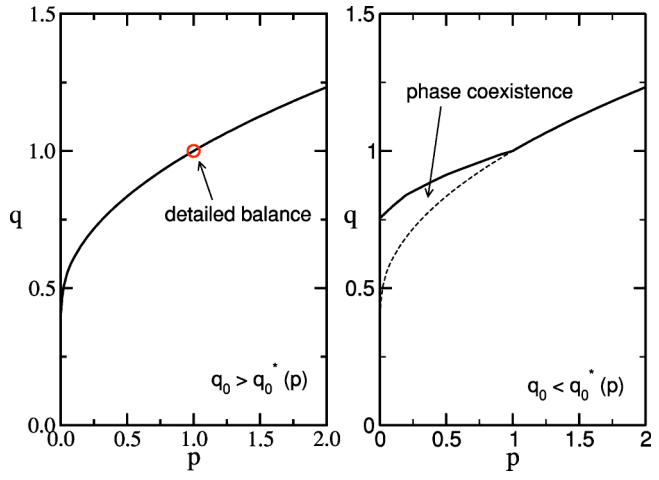

FIG. 7. Phase diagram of the RSOS model for $q_0 > q_0^*(p)$ (left panel) and $q_0 < q_0^*(p)$ (right panel). The phase transition takes place along the full line, while the dashed line in the right panel marks the lower border of the phase-coexistence region (see text).

$$\dot{h} = D\nabla^2 h + \nu(\nabla h)^2 - \frac{\partial V}{\partial h} + \zeta. \tag{11}$$

Here $h \equiv h(x,t)$ indicates the height of the interface on the substrate, while $\zeta \equiv \zeta(x,t)$ is the noise term δ-correlated in space and time, $\langle \zeta(x,t)\zeta(x',t')\rangle \sim \delta(x-x')\delta(t-t')$. The coefficient ν of the nonlinear term is positive for $p > 1$ and negative for $p < 1$. Detailed balance holds only for $p = 1$. This special case can be solved exactly [28]: its critical properties are described by an Edwards-Wilkinson (EW) equation [29] ($\nu = 0$) equipped with the interface-substrate interaction potential.

Decreasing q_0 amounts to increasing the attractive interaction between interface and substrate. It has been observed that, below a certain threshold, the continuous transition may turn into a discontinuous one [3]. For instance, at $p = 1$ it was shown analytically that for $q_0 < 2/3$ the transition becomes first order, while the transition point remains located at $q = 1$. Numerical simulations provide very clear evidence that this scenario extends to the nonequilibrium case $p > 1$: the phase-transition line is still independent of q_0 (see Fig. 7), while for q_0 smaller than a threshold value $q_0^*(p)$, the transition becomes first order. In this case, the interface dynamics is quite different from the one described by our model. In fact, direct inspection of the interface dynamics in the active phase close to criticality shows that there is a non-negligible probability for the interface to return in contact with the substrate not only at the domain boundaries of inactive domains, but also inside these domains. In DP jargon, this amounts to saying that an inactive site may become active even without being in contact with an active site. This excludes any direct relation with the model of a generalized contact process with long-range interactions introduced in Sec. II.

Conversely, for $0 < p < 1$, the dynamics appears to be strongly related to that of our model. For sufficiently small q_0 [i.e., q_0 smaller then a threshold $q_0^*(p)$], a region in the (p,q) plane arises, where the pinned and the unbound phases coexist [see Fig. 7(b)]. In this region, an unbound interface moves away from the substrate and never binds back. On the other hand, a bound interface remains bound for macroscopically long times and, in the thermodynamic limit, will never detach from the substrate. Only if the growth rate q is increased beyond a new critical value, a depinning transition

takes place at the upper border line of the phase-coexistence region [see Fig. 7(b)]. It is the nature of this transition that we want to investigate here.

As explained in [3,4], the stability of the pinned (active) phase in the coexistence region is ensured by the negative sign of the nonlinear coefficient ν: when a large detached island forms, it grows quickly, acquiring a triangular shape with a given slope and eventually shrinks from the edges with constant velocity. Because of the triangular shape of the interface, the probability of returning to the substrate at some point far from the edges of the island is exponentially suppressed. Therefore, it is reasonable to conjecture that for $p<1$ the model belongs to the class of contact processes.

Numerical simulations of the RSOS model dynamics (not shown here) seem to indicate that also in this case the depinning transition may become discontinuous for $q_0<q_0^*(p)$. For $p=0$ and, correspondingly, $q_0<q_0^*(0)\approx 0.399$, we know that it is of DP type [3]. On the other hand, the DP scaling regime becomes transparent only after a transient time that, for $q_0=0.35$ is on the order of 10^4 units. Although the crossover becomes practically unobservable for yet smaller q_0 values, there are compelling reasons to believe that this regime is eventually attained [3].

In order to shed some light about the order of the transition for $0<p<1$, we have tested whether effective long-range effects spontaneously emerge as a result of the RSOS microscopic dynamics. In practice, we have measured the effective activation rate $\bar{\lambda}(\ell)=N_a(\ell)/N_b(\ell)$ at the border of depinned islands of size ℓ ($N_b(\ell)$ is the number of times a depinned site at the border of an inactive island of size ℓ has been selected in the stationary regime and $N_a(\ell)$ is the number of times the selected site is immediately pinned). Data has been obtained by averaging over time, space and different realizations (typically, 100) for large lattices ($L=10^5$) and close to the transition line.

We have first analyzed the nature of the transition close to $p=0$. In Fig. 8 we display the results obtained at the transition points $T_0=(p=0,q_0=0.2,q=0.755)$ and $T_1=(p=0.01,q_0=0.2,q=0.76)$. The activation rate $\bar{\lambda}(\ell)$ is reported after subtracting its estimated asymptotic value $\bar{\lambda}_\infty=\lim_{\ell\to\infty}\bar{\lambda}(\ell)$. In both cases, it is found that it converges towards $\bar{\lambda}_\infty$ faster than $1/\ell$, although no precise estimate of the scaling rate can be obtained. Altogether, these results suggest that DP critical properties persist also for small values of p and $q_0<q_0^*(p)$. On the other hand, for $p=0.2$ and $q_0<q_0^*(0.2)=0.515\ldots$ our findings are suggestive of a first-order phase transition (and thus, in agreement with previous findings [3]). In fact, from Fig. 9, we see that in both points $T_2=(p=0.2,q_0=0.4,q=0.70)$, and $T_3=(p=0.2,q_0=0.3,q=0.745)$ the activation rate is found to converge slower than $1/\ell$, although the actual value of the exponent $\sigma<1$ appears to depend on the parameter. According to the criterion introduced in Sec. II, these results are therefore compatible with a discontinuous phase transition.

As a result, we can attribute the seemingly first-order nature of the transition to the existence of effective long-range interactions. Nevertheless, it remains to be proved whether the slow dependence of the activation rate on the window size is just a finite-size effect or holds fot arbitrarily large distances. Here below we present an argument supporting the former hypothesis.

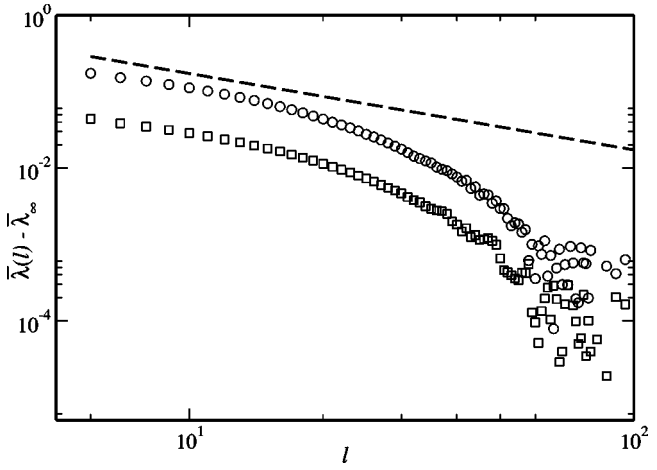

FIG. 8. RSOS model—decay of the activation rate (see text) as a function of size of depinned islands. Numerical simulation have been performed close to the special point $p=0$. Circles refer to $T_0=(p=0,q_0=0.2,q=0.755)$, while squares to $T_1=(p=0.01,q_0=0.2,q=0.76)$. The graph is plotted in a doubly logarithmic scale. For the sake of clarity the data are shifted vertically by an arbitrary value.

In order to clarify this point we move progressively away from the equilibrium case. For $p=1$, it is known that the transition is discontinuous [4] (for small enough q_0) and that the dynamics of the free interface is asymptotically described by the EW equation. When p is lowered below 1, the only relevant difference that is expected to occur is a crossover in the free interface dynamics from and EW to a KPZ regime above some critical length ℓ_c [32,33]. It is thus natural to conjecture that as long as the dynamics of the bound inter-

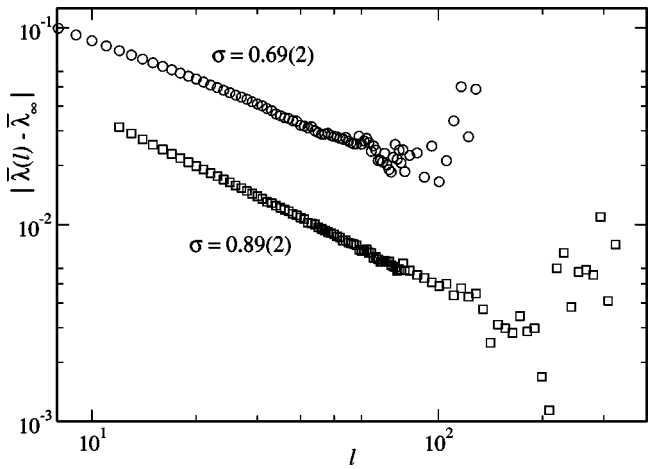

FIG. 9. RSOS model—decay of the activation rate (see text) as a function of size of depinned islands. Numerical simulation have been performed at the critical points $T_2=(p=0,q_0=0.4,q=0.70)$ (squares) and $T_3=(p=0.2,q_0=0.3,q=0.745)$ (circles). The graph is plotted in a doubly logarithmic scale. Data at $p=0.3$ show a power law decay with an exponent $\sigma=0.69(2)$, while $q_0=0.4$ data decays with an exponent $\sigma=0.89(2)$.

face remains insensitive to such differences, the scenario should not change. Here below we argue that this occurs for system sizes smaller than some length L_c that can be extremely large.

The best way to characterize the above mentioned crossover is by monitoring the width $w(t)=\langle\langle(h-\langle h\rangle)^2\rangle\rangle^{1/2}$ of an initially flat (and free) KPZ interface. In fact, after an initial growth as $t^{1/4}$ (in agreement with EW equation), at some time t_c, $w(t)$ crosses over towards a behavior of the type $t^{1/3}$ and eventually saturates because of the finite length of the system. In other words, for times smaller than t_c the interface behaves in the same way as in an equilibrium regime. Since the stationary profile of an interface is a diffusive random walk, one can safely assume that

$$w_\infty = \lim_{t\to\infty} w(t) = k\sqrt{L}, \tag{12}$$

where L is the interface length. By extending the above relation to finite times, one can interpret it as the definition of the effective scale $L(t)$ that is resolved at time t. For instance, $\ell_c=\langle w(t_c)/k\rangle^2$ is the minimal length of a free interface that allows observing a crossover to the KPZ scaling behavior.

Within the context of a bounded interface, this implies that deviations from equilibrium are observable only in those depinned islands of length $\ell>\ell_c$. Accordingly, the problem of determining the minimal length to observe deviations from the equilibrium scenario amounts to estimating the probability for a suitably large depinned islands to arise. At equilibrium, the theoretical analysis developed in [4] has revealed that when the transition is discontinuous (the scenario we are interested in), the interface is exponentially localized at the substrate, i.e., the probability to find large values of the height h scales as $P(h)\simeq\exp(-h/h_0)$. It is quite plausible to assume, and we have indeed numerically verified, that as long as deviations from equilibrium are not detectable, the exponential decay survives also for $p<1$. Since we have also seen that depinned islands have an approximately triangular shape, this means that also island lengths are exponentially distributed $P(\ell)\simeq\exp(-\ell/\ell_0)$, where ℓ_0 is proportional to h_0, the proportionality constant being related to the slope of such islands.

As a result, the probability that at least a given island reaches the size ℓ_c is proportional to $\exp(-\ell_c/\ell_0)$. In a large but finite system of size L, this may happen independently at different places. Hence the first large island would appear in a typical time $\tau\simeq\ell_c/L\exp(\ell_c/\ell_0)$. Accordingly, the minimum system size guaranteeing that such islands are observed with nonnegligible probability and dominate the wetting dynamics is

$$L_c = \ell_c \exp(\ell_c/\ell_0). \tag{13}$$

Therefore, as long as the (effective) size remains smaller than L_c, a seemingly first-order transition is observed. Beyond L_c, in the fully nonlinear regime, several theoretical and numerical studies of different models [8,14] suggest that a DP behavior sets in. As a result, we expect in particular that the distribution of depinned islands crosses over from an exponential to a power-law distribution. However, given the exponential dependence of L_c on ℓ_c, it may happen that the crossover length is so large that the asymptotic regime is practically unobservable. Note also that L_c diverges exponentially as $p\to1$, thus approaching the equilibrium point.

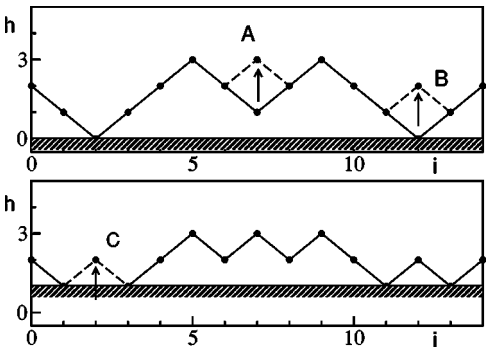

FIG. 10. Updating rule of the SSW model. The full line represents the interface, while the shaded area represents the wall. Dashed segments indicate interface growth occurring in randomly chosen local minima (see A and B in upper panel) and in all sites located below the wall after it has been shifted upwards by one.

Numerical simulations performed at the transition point $T_1=(p=0.01,q=0.76,q_0=0.2)$ indicate that $k=0.179(1)$ and $w(t_c)=2.9(2)$, yielding the estimate $\ell_c\approx260$. On the other hand, direct computations of the island sizes indicate that $\ell_0\sim O(10^2)$, yielding $L_c\approx3.5\times10^3$ and altogether confirming that the crossover towards DP can be observed, as we actually do. Moreover, at the transition point $T_2=(p=0.2,q=0.70,q_0=0.4)$, we find $k=0.204(1)$ and $w(t_c)=5.7(2)$, while ℓ_0 is almost unchanged, yielding the estimates $l_c\approx780$ and $L_c\approx2\times10^6$. Consistently, no indication of the crossover has been observed up to times on the order $O(10^6)$ and system sizes of length $O(10^5)$.

B. Single-step-with-wall model

The second model we have tested is the so-called "single-step-with-wall" (SSW) model [8], a variant of the well known single-step model introduced in Refs. [30,31]. Here, the growing interface is described by a set of integer heights h_i at site i of a one-dimensional lattice of length L with periodic boundary conditions. In this model, the "continuity" restriction $|h_i-h_{i+1}|=1$ plays the role the RSOS constraint. An upward-moving wall is located at some integer height $h_w(t)$, below the interface. It moves with velocity v_w, thus pushing the interface which cannot be overtaken by the wall. Moreover, in analogy with the RSOS model, the interface is also attracted by the wall. The model evolves by random-sequential dynamics, i.e., at each time step $dt=1/L$, a site i is chosen at random. If the interface has a local minimum at site i (namely $h_i<h_{i\pm1}$), the height h_i is increased by two units with probability 1 if $h_i>h_w$, or with probability $(1-q)$ whenever the interface is pinned to the wall ($h_i=h_w$). Since $0<q<1$, an effective attractive force is introduced between the wall and the interface. After $n_w=L/v_w$ time steps, the wall is moved upward by one unit, while the height of all interfacial sites that would be overtaken by the wall is increased by two units (e.g., see Fig. 10). On the basis of these microscopic update rules, one can easily infer that the

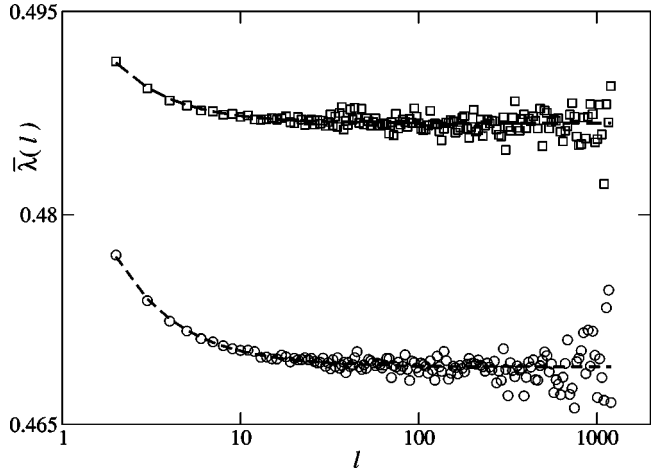

FIG. 11. SSW model—Double logarithmic plot of the activation rate $\overline{\lambda}(\ell)$ as a function of the size ℓ of depinned islands. Numerical simulations have been performed at criticality: circles correspond to $q=0.7$, $v_w=0.4297$ while squares to $q=0.8$, $v_w=0.3489$. Dashed lines mark our best fits with Eq. (1), rendering respectively the estimates $\sigma=1.20(5)$ and $\sigma=1.30(5)$.

velocity of the free interface is $1/2$ in the thermodynamic limit.

The phase diagram of the SSW model is controlled by two parameters: $v_w-1/2$, i.e., the relative velocity of the wall to the free interface, and q, i.e., the "stickiness" of the wall. By decreasing v_w, a phase transition from a pinned to a depinned phase is observed. Numerical analysis and analytical arguments [8] show that when $q<q^*=0.444\ldots$ such a transition takes place at $v_w^c=1/2$, i.e., when the relative velocity of the wall with respect to the free-interface changes its sign. The transition in this part of the phase diagram is continuous and belongs to the MN universality class. For $q>q^*$ the effective attractive force binds the interface to the wall and a continuous DP phase transition takes place at a critical value $v_w^c(q)<1/2$.

In order to estimate the role of long range interactions in the critical dynamics of the SSW model, we have measured also in this case the effective activation rate $\overline{\lambda}(\ell)$. Since the time interval $1/v_w$ between two consecutive wall moves can be regarded as the natural time scale of the SSW model, we measured the pinning rate at time $t_m=m/v_w$ $(m=1,2,\ldots)$ for an initially depinned site located at the border of a depinned island of size ℓ at time t_{m-1}. Averages have been taken over time, space and different ensemble realizations (typically 100) for large enough systems ($L=10^5$) close to the DP critical line, at $q=0.7$ and $q=0.8$. Our results are shown in Fig. 11, where numerical data has been fitted with $\overline{\lambda}(\ell)=\lambda(1+a/\ell^\sigma)$. There is evidence that $\overline{\lambda}(\ell)$ decays to a constant faster than $1/\ell$ with exponents $\sigma=1.20(5)$ (for $q=0.7$) and $\sigma=1.30(5)$ ($q=0.8$). According to our predictions based on the behavior of the generalized contact process, this implies a continuous DP transition, which indeed has been observed in Ref. [8].

Unlike the RSOS model discussed in the previous section, here in the SSW model the DP behavior is observed at small length scales, and is thus accessible in numerical studies of finite systems. The reason is that the SSW model is designed in a way that the nonlinearity is maximal [32], and hence the crossover length ℓ_c is fixed and of order 1. On the other hand, in the RSOS model the nonlinearity depends on the growth parameter p, with a diverging crossover length as $p\to 1$.

V. CONCLUSIONS

In this work we examined a possible connection between wetting phenomena and contact processes. Inspired by the puzzling richness of the phase diagrams found in various models, we introduced a generalized contact process with the goal of capturing both the DP and first-order transitions observed in one-dimensional nonequilibrium wetting transitions occurring at nonzero interface velocity. The element of novelty distinguishing our model from standard contact processes consists in an algebraic dependence of the activation rate on the length ℓ of the depinned island containing the inactive site $\overline{\lambda}(\ell)\simeq\lambda(1+a/\ell^\sigma)$.

A mean-field analysis predicts that when $\sigma>1$, the model exhibits a continuous phase transition characterized by DP critical exponents. In other words, the algebraic decay of the interactions is not so long-range as to alter DP critical properties. Conversely, for $0<\sigma<1$ the phase transition turns to a first-order one. Numerical simulations confirm the mean-field predictions, which are found to provide also an accurate estimate for the critical value σ_c, separating the two different regimes: $\sigma_c=1.0\pm0.1$, from numerics. By directly measuring the effective activation rate $\overline{\lambda}(\ell)$ in nonequilibrium wetting processes, one can use the estimate of the exponent σ as a practical tool for probing the nature of a wetting transition.

To apply the insight gained from the DP process to wetting phenomena, we considered in this paper two wetting models, RSOS and SSW (see Sec. IV). Previous numerical studies of the latter model [8] indicate that for a sufficiently strong attractive force of the substrate (wall), the wetting transition is DP. Our criterion confirms these results, since the effective activation rate is indeed found to converge faster than $1/\ell$ to its asymptotic value. The phase diagram of the RSOS model on the other hand is known to be more complicated, since it contains both first order and continuous wetting transitions [3]. Our analysis suggests that for $p<1$ the entire phase-transition line located at the upper border of the coexistence region should asymptotically belong to the DP universality class. However, the finite-size behavior depends on two length scales: (i) the crossover ℓ_c between EW and KPZ roughening behavior; (ii) the effective size ℓ_0 of depinned islands. As long as the system size is smaller then $L_c=\ell_c\exp(\ell_c/\ell_0)$ and islands of size larger than ℓ_c are not significatively generated, numerical simulations can only provide evidence of a first-order transition. A qualitative phase diagram for the RSOS model in the case $q_0<q_0^*(p)$ is sketched in Fig. 12.

It is interesting to compare our results with those recently obtained for the KPZ equation with an attracting hard core potential [14]: $\dot{h}=D\nabla^2h+\nu(\nabla h)^2-V'(h)+\zeta$. Although the au-

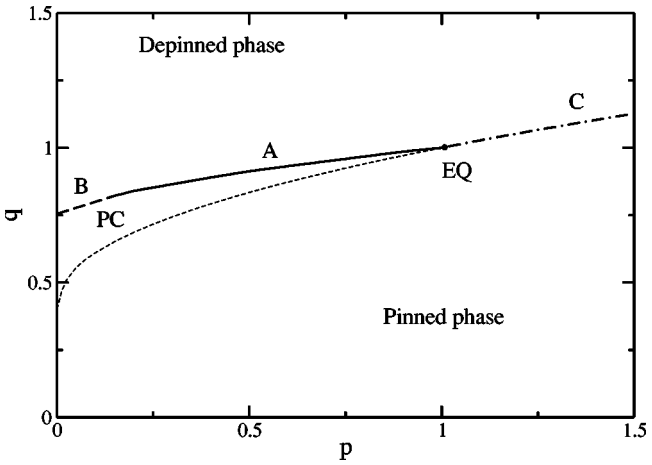

FIG. 12. Qualitative "finite-size" phase diagram of the RSOS model in the case $q_0 < q_0^*(p)$. We expect the entire transition line for $p<1$ to belong to the DP universality class. The full line (line A) on the left of the equilibrium point (EQ) marks a region in which the transient first-order behavior is the only numerically accessible one, with a crossover length L_c to DP behavior which diverges as $p \to 1$. On the other hand, also systems of moderate size can exhibit a DP critical behavior over the dashed line (B). The dot dashed transition line (C) on the right of the equilibrium point is genuinely first order (see text). Finally, the light dashed line where the free interface velocity changes sign marks the lower boundary of the phase-coexistence region (PC).

thors do not exclude the possibility that the entire transition line belongs to the DP universality class, their numerical analysis revealed both a first order and a DP wetting transition. In particular, DP critical behavior has been observed upon decreasing the attractive force and reducing by a factor ten *both* the diffusion (D) and the nonlinear (ν) term with respect to the parameter values corresponding to a first-order phase transition. While decreasing the attractive force is obviously expected to increase the "cutoff" scale ℓ_0, the latter change decreases the length scale ℓ_c by the same factor ten. The crossover length from EW to KPZ roughening is in fact known to scale as $D^3/(\nu^2\Gamma)$ [33], where Γ is the amplitude of the Gaussian white noise ζ. This indeed suggests that the DP (first-order) behavior has been observed in a parameter range where the crossover scale L_c [as defined by Eq. (13)] is numerically accessible (unaccessible), as suggested in the present study. This work helps to better understand the part of the wetting phase diagram characterized by a negative coefficient of the KPZ nonlinearity, which is precisely the part related with the complete synchronization transition (in the case of an MN phase transition, this relation can be made explicit by use of the Cole-Hopf transformation [34]). In particular our analysis indicates that the dynamical details of the system may induce a seemingly first order phase transition which lasts over exponentially long time and space scales, effectively suppressing DP critical properties.

Finally, we wish to comment about the transition line for $p>1$ (where the coefficient of the KPZ nonlinearity is positive). For $q_0 < q_0^*(p)$, the transition is known to be first order, but it occurs when the interface velocity changes sign [3], i.e., there is no region of phase coexistence. Accordingly, depinned islands are no longer characterized by a triangular shape and interface fluctuations may easily give rise to the pinning of inactive sites far away from the active ones; in other words, the analogy with contact process is seemingly lost. It would be interesting to investigate whether the inclusion of some sort of "spontaneous" nucleation of active sites can eventually account for the scenario observed for $p>1$. A further open problem is the crossover from the DP to the MN universality class, which takes place in both RSOS and SSW models. The study of this crossover would require the inclusion of nucleation of active sites in the interior of inactive domains. An appropriate generalization of the DP model including such processes could yield useful insight onto the crossover phenomena taking place in nonequilibrium wetting.

ACKNOWLEDGMENTS

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